Circle theorems

Key points
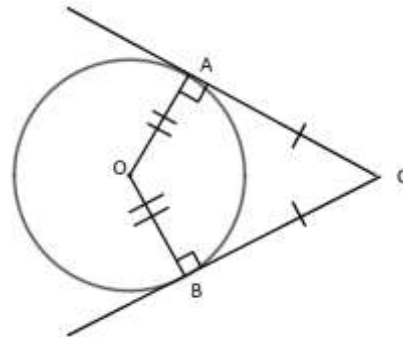
- A chord is a straight line joining two points on the circumference of a circle.
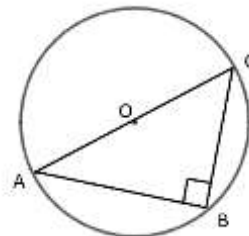 So AB is a chord.
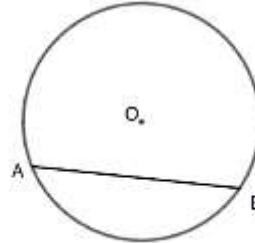
- A tangent is a straight line that touches the circumference of a circle at only one point.
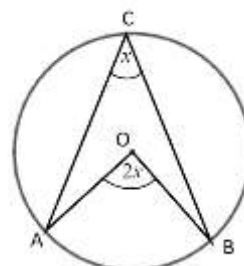 The angle between a tangent and the radius is 90°.
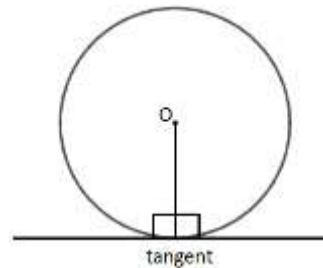
- Two tangents on a circle that meet at a point outside the circle are equal in length.
  So AC = BC.
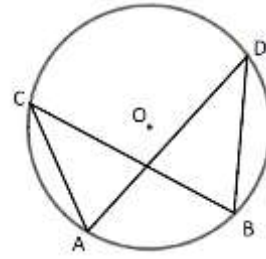

- The angle in a semicircle is a right angle.
  So angle ABC = 90°.

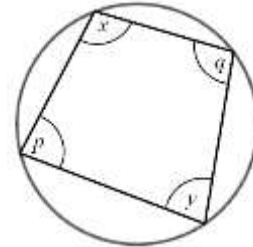- When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.
 So angle AOB = 2 × angle ACB.

- Angles subtended by the same arc at the circumference are equal. This means that angles in the same segment are equal.
  So angle ACB = angle ADB and angle CAD = angle CBD.

- A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle. Opposite angles in a cyclic quadrilateral total 180°.
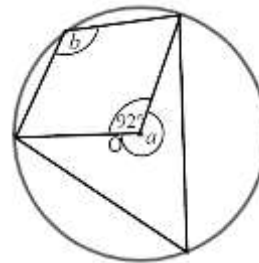 So $x + y = 180°$ and $p + q = 180°$.

- The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem.
  So angle BAT = angle ACB.
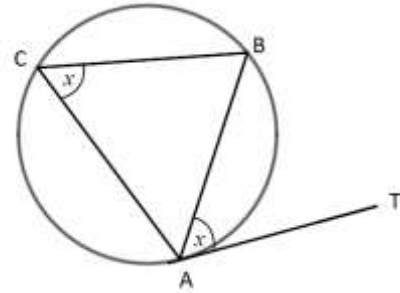
Examples

Example 1   Work out the size of each angle marked with a letter.
Give reasons for your answers.

Angle $a = 360° − 92°$ $= 268°$ as the angles in a full turn total 360°.	**1** The angles in a full turn total 360°.
Angle $b = 268° ÷ 2$ $= 134°$ as when two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.	**2** Angles a and b are subtended by the same arc, so angle b is half of angle a.

Example 2   Work out the size of the angles in the triangle.
Give reasons for your answers.

Angles are 90°, 2*c* and *c*.	**1** The angle in a semicircle is a right angle.
90° + 2*c* + *c* = 180°	**2** Angles in a triangle total 180°.
90° + 3*c* = 180°	**3** Simplify and solve the equation.
3*c* = 90°	
c = 30°	
2*c* = 60°	
The angles are 30°, 60° and 90° as the angle in a semi-circle is a right angle and the angles in a triangle total 180°.	

Example 3   Work out the size of each angle marked with a letter.
Give reasons for your answers.

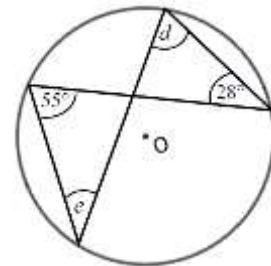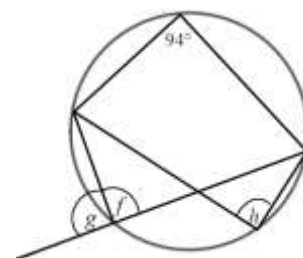| Angle *d* = 55° as angles subtended by the same arc are equal.	**1** Angles subtended by the same arc are equal so angle 55° and angle *d* are equal.
Angle *e* = 28° as angles subtended by the same arc are equal.	**2** Angles subtended by the same arc are equal so angle 28° and angle *e* are equal.

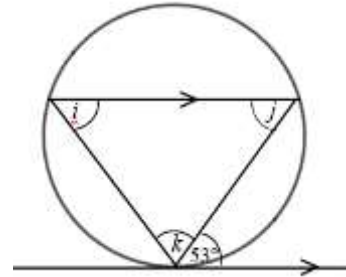**Example 4**   Work out the size of each angle marked with a letter.
Give reasons for your answers.

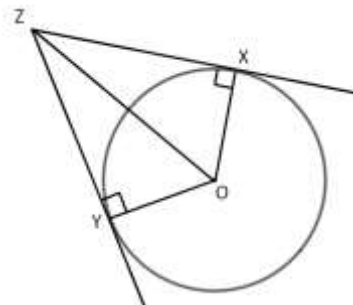| Angle *f* = 180° − 94° = 86° as opposite angles in a cyclic quadrilateral total 180°.	**1** Opposite angles in a cyclic quadrilateral total 180° so angle 94° and angle *f* total 180°.
	(continued on next page)

Angle $g = 180° − 86°$	**2** Angles on a straight line total 180°
$= 84°$ as angles on a straight line total 180°.	so angle f and angle g total 180°.
Angle h = angle $f = 86°$ as angles subtended by the same arc are equal.	**3** Angles subtended by the same arc are equal so angle f and angle h are equal.

Example 5 Work out the size of each angle marked with a letter.
Give reasons for your answers.

Angle $i = 53°$ because of the alternate segment theorem.	**1** The angle between a tangent and chord is equal to the angle in the alternate segment.
Angle $j = 53°$ because it is the alternate angle to 53°.	**2** As there are two parallel lines, angle 53° is equal to angle j because they are alternate angles.
Angle $k = 180° − 53° − 53°$ $= 74°$ as angles in a triangle total 180°.	**3** The angles in a triangle total 180°, so $i + j + k = 180°$.

Example 6 XZ and YZ are two tangents to a circle with centre O.
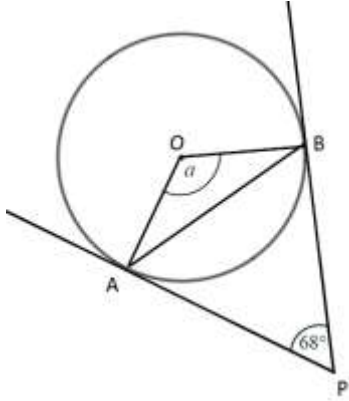Prove that triangles XZO and YZO are congruent.

| Angle OXZ = 90° and angle OYZ = 90° as the angles in a semicircle are right angles.	For two triangles to be congruent you need to show one of the following.
OZ is a common line and is the hypotenuse in both triangles.	• All three corresponding sides are equal (SSS). • Two corresponding sides and the included angle are equal (SAS).
OX = OY as they are radii of the same circle.	• One side and two corresponding angles are equal (ASA).
So triangles XZO and YZO are congruent, RHS.	• A right angle, hypotenuse and a shorter side are equal (RHS).

Practice

1 Work out the size of each angle marked with a letter.
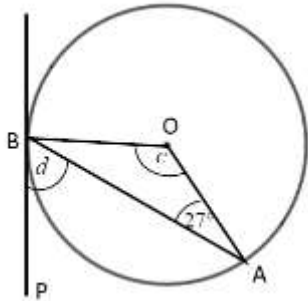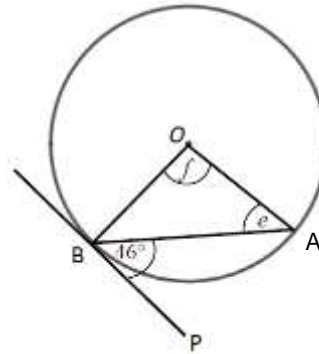Give reasons for your answers.

a

b
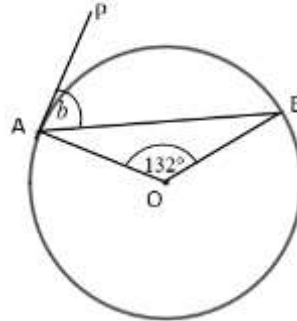
c

d

e

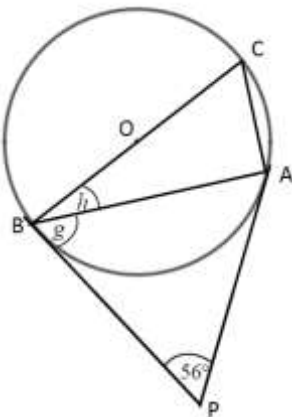**2** Work out the size of each angle marked with a letter.
Give reasons for your answers.

a

b

c

Hint

The reflex angle at point O and angle *g* are subtended by the same arc. So the reflex angle is twice the size of angle *g*.

d

Hint

Angle 18° and angle *h* are subtended by the same arc.

3   Work out the size of each angle marked with a letter.
Give reasons for your answers.

a

b
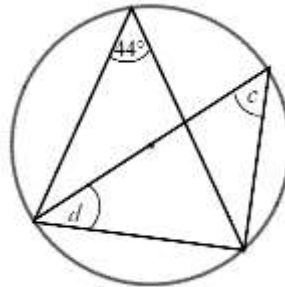
Hint

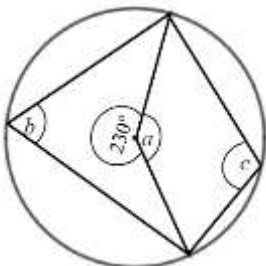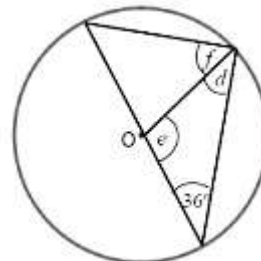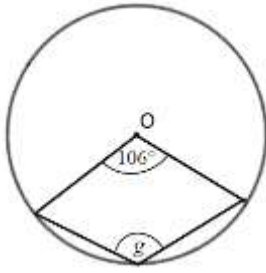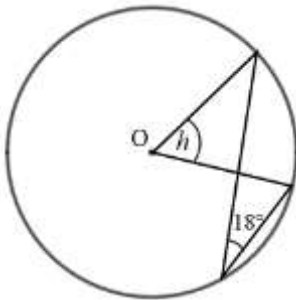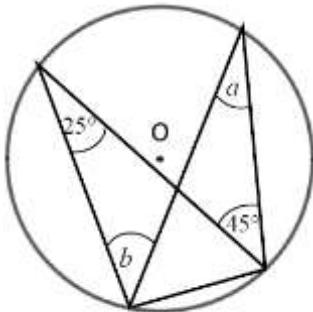One of the angles is in a semicircle.

c
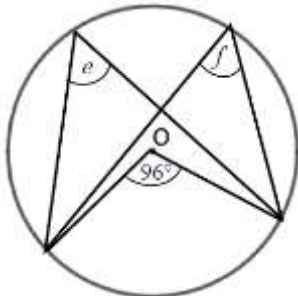
d
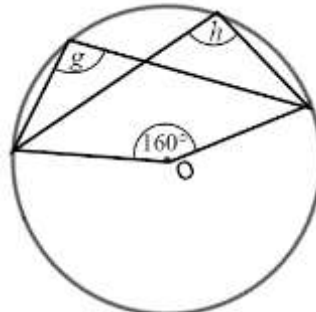
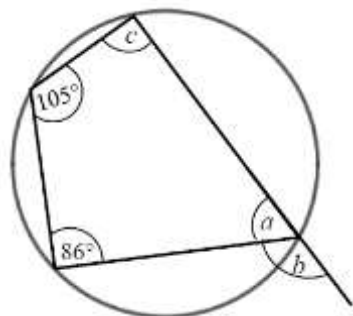**4**   Work out the size of each angle marked with a letter.
          Give reasons for your answers.

a

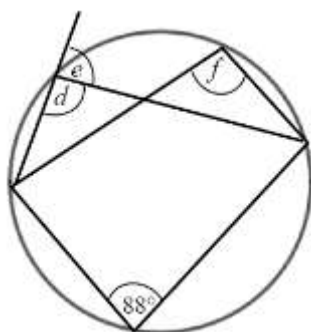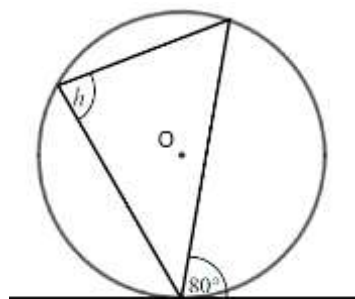

> **Hint**
>
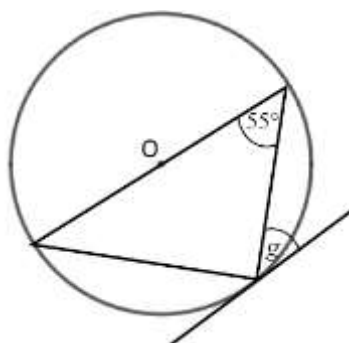> An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

b   **c**

d

> **Hint**
>
> One of the angles is in a semicircle.

Extend

5   Prove the alternate segment theorem.

Answers

1 **a** $a = 112°$, angle OAP = angle OBP = 90° and angles in a quadrilateral total 360°.

    b $b = 66°$, triangle OAB is isosceles, Angle OAP = 90° as AP is tangent to the circle.

    c $c = 126°$, triangle OAB is isosceles.
        $d = 63°$, Angle OBP = 90° as BP is tangent to the circle.

    d $e = 44°$, the triangle is isosceles, so angles e and angle OBA are equal. The angle OBP = 90°
           as BP is tangent to the circle.
        $f = 92°$, the triangle is isosceles.

    e $g = 62°$, triangle ABP is isosceles as AP and BP are both tangents to the circle.
        $h = 28°$, the angle OBP = 90°.

2 **a** $a = 130°$, angles in a full turn total 360°.
        $b = 65°$, the angle at the centre of a circle is twice the angle at the circumference.
        $c = 115°$, opposite angles in a cyclic quadrilateral total 180°.

    b $d = 36°$, isosceles triangle.
        $e = 108°$, angles in a triangle total 180°.
        $f = 54°$, angle in a semicircle is 90°.

    c $g = 127°$, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle
           at the circumference.

    d $h = 36°$, the angle at the centre of a circle is twice the angle at the circumference.

3 **a** $a = 25°$, angles in the same segment are equal.
        $b = 45°$, angles in the same segment are equal.

    b $c = 44°$, angles in the same segment are equal.
        $d = 46°$, the angle in a semicircle is 90° and the angles in a triangle total 180°.

    c $e = 48°$, the angle at the centre of a circle is twice the angle at the circumference.
        $f = 48°$, angles in the same segment are equal.

    d $g = 100°$, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle
           at the circumference.
        $h = 100°$, angles in the same segment are equal.

4 **a** $a = 75°$, opposite angles in a cyclic quadrilateral total 180°.
        $b = 105°$, angles on a straight line total 180°.
        $c = 94°$, opposite angles in a cyclic quadrilateral total 180°.

    b $d = 92°$, opposite angles in a cyclic quadrilateral total 180°.
        $e = 88°$, angles on a straight line total 180°.
        $f = 92°$, angles in the same segment are equal.

    c $h = 80°$, alternate segment theorem.

    d $g = 35°$, alternate segment theorem and the angle in a semicircle is 90°.

5 Angle BAT = x.

Angle OAB = $90° − x$ because the angle between the tangent and the radius is 90°.

OA = OB because radii are equal.

Angle OAB = angle OBA because the base of isosceles triangles are equal.

Angle AOB = $180° − (90° − x) − (90° − x) = 2x$ because angles in a triangle total 180°.

Angle ACB = $2x ÷ 2 = x$ because the angle at the centre is twice the angle at the circumference.